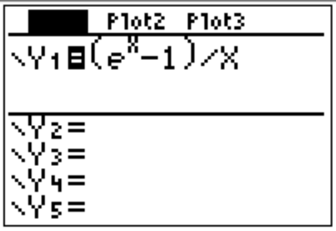
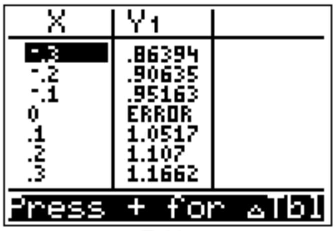
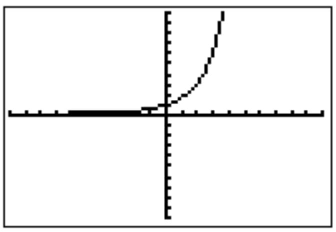


Equation



Table


Graph

# Derivative of $e^x$

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$\lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$

$$\lim_{h\to 0} e^x \frac{(e^h - 1)}{h}$$

$$\lim_{h\to 0} e^x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

**EXAMPLE 1 Using the Formula**

Find $dy/dx$ if $y = e^{(x+x^2)}$.

$$\frac{dy}{dx} = e^{(x+x^2)} \cdot (2x+1)$$

# Derivative of $a^x$

constant

$$a^x = e^{x \underline{\ln a}}.$$

$$e^{x \ln a} = e^{\ln(a^x)} = a^x$$

$$\frac{d}{dx} e^{x \ln a} = e^{\ln a x} \cdot \ln a$$

$$= a^x \ln a$$

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}.$$

**EXAMPLE 2 Reviewing the Algebra of Logarithms**

At what point on the graph of the function $y = 2^t - 3$ does the tangent line have slope 21?

$$\frac{dy}{dt} = 2^t \ln 2$$

$$21 = 2^t \ln 2$$

$$\frac{21}{\ln 2} = 2^t$$

$$\log_2 \frac{21}{\ln 2} = t$$

$$t \approx 4.921$$

point:
$(4.921, 27.297)$

**EXPLORATION 1** **Leaving Milk on the Counter**

A glass of cold milk from the refrigerator is left on the counter on a warm summer day. Its temperature $y$ (in degrees Fahrenheit) after sitting on the counter $t$ minutes is

$$y = 72 - 30\underline{(0.98)}^t.$$

Answer the following questions by interpreting $y$ and $dy/dt$.

1. What is the temperature of the refrigerator? How can you tell? $42°F$ at $t=0$
2. What is the temperature of the room? How can you tell? $72°F$
3. When is the milk warming up the fastest? How can you tell? when it comes out of the fridge
4. Determine algebraically when the temperature of the milk reaches 55°F.
5. At what rate is the milk warming when its temperature is 55°F? Answer with an appropriate unit of measure.

derivative

$$\frac{dy}{dx} = -30(.98)^t \cdot \ln .98$$
$$\approx 0.606(.98)^t$$

4 $$55 = 72 - 30(.98)^t$$
$$\frac{17}{30} = (.98)^t$$
$$\log_{.98}\left(\frac{17}{30}\right) = t$$
$$t \approx 28.114 \text{ minutes}$$

5. $$-30(.98)^{\text{ans.}} \cdot \ln .98$$
$$\approx .343\ °F/\text{min}$$

## Derivative of ln *x*

$$y = \ln x$$

$$y = \log_e x$$

Same

$$e^y = x$$

$$\frac{d}{dx}: \quad e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}.$$

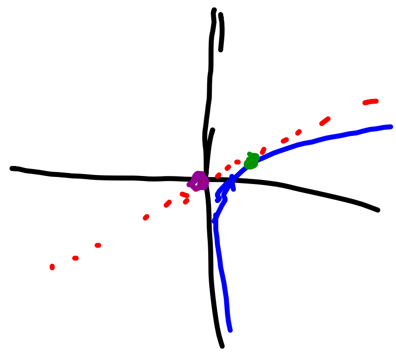
**EXAMPLE 3 A Tangent through the Origin**

A line with slope $m$ passes through the origin and is tangent to the graph of $y = \ln x$. What is the value of $m$?

$$\frac{dy}{dx} = \frac{1}{x}$$

$(a, \ln a)$ $(0,0)$

point of tangency



$$m = \frac{\ln a - 0}{a - 0}$$

$$= \frac{\ln a}{a}$$

$$\frac{\ln a}{a} = \frac{1}{a}$$

$$\ln a = 1$$

$$a = e$$

$$\text{slope} = \frac{1}{a} = \frac{1}{e}$$

# Derivative of $\log_a x$

$$\log_a x = \frac{\ln x}{\ln a}.$$

$\frac{1}{\ln a}$ → constant

$$= \frac{1}{\ln a} \cdot \ln x$$

$$\frac{d}{dx} \log_a x = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$= \frac{1}{x \ln a}$$

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}.$$

**EXAMPLE 4** **Going the Long Way with the Chain Rule**

Find $dy/dx$ if $y = \log_a(a^{\sin x})$

$$\frac{dy}{dx} = \frac{1}{\ln a\,(a^{\sin x})} \cdot a^{\sin x} \cdot \ln a \cdot \cos x$$

$$= \cos x$$

# Power Rule for Arbitrary Real Powers

$$x^n = e^{n \ln x}.$$

**RULE 10 Power Rule for Arbitrary Real Powers**

If $u$ is a positive differentiable function of $x$ and $n$ is any real number, then $u^n$ is a differentiable function of $x$, and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

**(a)** If $y = x^{\sqrt{2}}$,

$y' = \sqrt{2}\, x^{\sqrt{2}-1}$

If $y = (2 + \sin 3x)^{\pi}$

$y' = \pi (2 + \sin 3x)^{\pi - 1} \cdot (\cos 3x) \cdot 3$

$= 3\pi (2 + \sin 3x)^{\pi - 1} (\cos 3x)$

**EXAMPLE 6 Finding Domain**

If $f(x) = \ln(x - 3)$, find $f'(x)$. State the domain of $f'$.

domain of $f(x)$ : $x > 3$

$f'(x) = \frac{1}{x-3}$

domain of $f'(x) = (3, \infty)$

**EXAMPLE 7** **Logarithmic Differentiation**

Find $dy/dx$ for $y = x^x$, $x > 0$.

**EXAMPLE 8 How Fast does a Flu Spread?**

The spread of a flu in a certain school is modeled by the equation

$$P(t) = \frac{100}{1 + e^{3-t}},$$

where $P(t)$ is the total number of students infected $t$ days after the flu was first noticed. Many of them may already be well again at time $t$.

**(a)** Estimate the initial number of students infected with the flu.

**(b)** How fast is the flu spreading after 3 days? → derivative

**(c)** When will the flu spread at its maximum rate? What is this rate?

a) $P(0) = \dfrac{100}{1+e^{3}} \approx 5$ students

b) $100\left(1+e^{3-t}\right)^{-1}$

$P'(t) = -100\left(1+e^{3-t}\right)^{-2} \cdot \left(e^{3-t}\right)(-1)$

$P'(3) \approx 25$ students/day

c) $P'(t) = \dfrac{100e^{3-t}}{\left(1+e^{3-t}\right)^{2}}$

max at $\approx 3$ days